Transportation Problems

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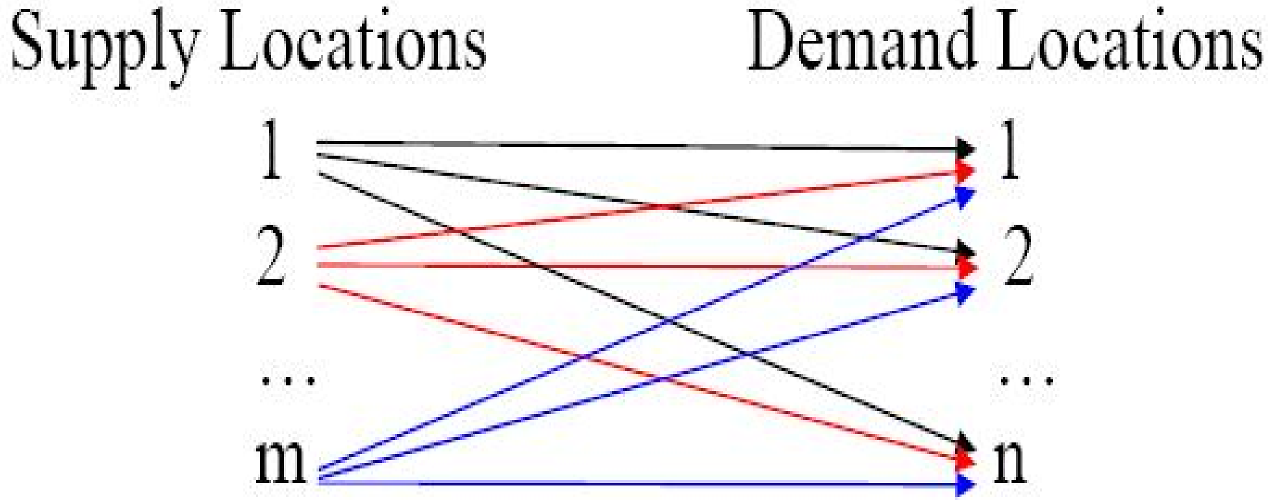
Description

A transportation problem basically deals with the problem, which aims to find the best way to fulfill the demand of *n* demand points using the capacities of *m* supply points. While trying to find the best way, generally a variable cost of shipping the product from one supply point to a demand point or a similar constraint should be taken into consideration.



Transportation model

- Companies produce products at locations called sources and ship these products to customer locations called destinations.

- Each source has a limited quantity that can ship and each customer destination must receive a required quantity of the product.

- Only possible shipments are those directly from a source to a destination.

The problems with the above characteristics are generally called “transportation problems”. These problems involve the shipment of a homogeneous product from a number of supply locations to a number of demand locations.

A typical transportation problem requires three sets of numbers:

- Capacities (or supplies)
  Indicate the most each plant can supply in a given time period.

- Demands (or requirements)
  They are typically estimated from some type of forecasting model. Often demands are based on historical customer demand data.

- Unit shipping (and possibly production) cost
  It is calculated through a transportation cost analysis.

- The transportation or shipping problem involves determining the amount of goods or items to be transported from a number of sources to a number of destinations.

- Usually the objective is to minimize total shipping costs or distances.

- Transportation problem is a specific case of Linear Programming problems and a special algorithm has been developed to solve it.

The problem:
Given needs at the demand locations, how should we take the limited supply at supply locations and move the goods. The objective is to minimize the total transportation cost.

Basic concept

- <u>Objective</u>: Minimize cost
- <u>Variables</u>: Quantity of goods shipped from each supply point to each demand point
- <u>Restrictions</u>:
  - Non negative shipments
  - Supply availability at each supply location
  - Demand need at each demand location

Transportation problems variables

Symbol	Variable
m	Sources (supply or production locations)
n	Destinations (demand or consumption locations)
a_i	Capacity (supply or production) of source i
b_j	Need (demand or consumption) of destination j
c_{ij}	Unit transportation cost from source i to destination j
x_{ij}	Quantity shipped from source i to destination j
C	Total transportation cost

Mathematic model

$$\min C = c_{11}x_{11} + c_{12}x_{12} + \dots + c_{1n}x_{1n} + c_{21}x_{22} + c_{22}x_{22} + \dots + c_{2n}x_{2n} + \dots\dots + c_{m1}x_{m1} + c_{m2}x_{m2} + \dots + c_{mn}x_{mn}$$

with

capacity constraints:

$$x_{11} + x_{12} + \dots + x_{1n} = a_1$$
$$x_{21} + x_{22} + \dots + x_{2n} = a_2$$
$$\dots\dots\dots$$
$$x_{m1} + x_{m2} + \dots + x_{mn} = a_m$$

needs constraints:

$$x_{11} + x_{21} + \dots + x_{m1} = b_1$$
$$x_{12} + x_{22} + \dots + x_{m2} = b_2$$
$$\dots\dots\dots$$
$$x_{1n} + x_{2n} + \dots + x_{mn} = b_n$$

and the non negativity constraints $x_{ij} \geq 0, \forall\ i,j$

Transportation problem standard model

$$\min \; C = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

with constraints

$$\sum_{j=1}^{n} x_{ij} = a_i$$

$$\sum_{i=1}^{m} x_{ij} = bj$$

and $x_{ij} \geq 0$,

where i = 1, 2, ..., m και j = 1, 2, ..., n

This problem has feasible solutions only if the total of the sources' capacities is equal to the total of destinations' needs, that is:

$$\sum_{j=1}^{n} a_i = \sum_{i=1}^{m} b_i = \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij}$$

Assumptions

- Unit transportation costs are independent of transported volume.
- Supply and demand are known and independent on price charged for the product.
- Unlimited transportation capacity to ship across any particular transportation route.
- A single commodity is transported.

Standard form of transportation tableau

i j	(1)	(2)	…	(n)	a_i
(1)	x_{11} c_{11}	x_{12} c_{12}	… …	x_{1n} c_{1n}	a_1
(2)	x_{21} c_{21}	x_{22} c_{22}	… …	x_{2n} c_{2n}	a_2
…	… …	… …	… …	… …	…
(m)	x_{m1} c_{m1}	x_{m2} c_{m2}	… …	x_{mn} c_{mn}	a_m
b_j	b_1	b_2		b_n	$\Sigma a_i = \Sigma b_j$

Transportation problem example

A mining company extracts gravel, the basic product it sells, from three mines, L1, L2 and L3. The weekly production of each mine is 75, 150 and 75 tones of gravel respectively. The gravel has to be transported to five main consumers, K1, K2, K3, K4 and K5 requiring for their needs 100, 60, 40, 75 and 25 tones of gravel per week respectively.

The problem that concerns the company's management is the minimization of the required cost for the transportation of the product to the consumers. For this purpose a detailed cost analysis was carried out which gave the results of the following table (the numbers denote the transportation cost in € per ton of gravel).

Cost table of gravel transportation

Consumers

Mines

	K_1	K_2	K_3	K_4	K_5
L_1	3	2	3	4	1
L_2	4	1	2	4	2
L_3	1	0	5	3	2

Methods for finding an initial basic feasible solution

1. Northwest corner method

2. Minimum cost method

3. Vogel method

Northwest corner method

1. The maximum possible quantity is assigned to the northwest (up left) cell depending on the supply and demand of the corresponding row or column. The supply of the row and the demand of the column are adjusted appropriately.

2. Either the row of which the supply is exhausted or the column of which the demand is satisfied is crossed out.

3. If all supplies are exhausted and all demands are satisfied then END, otherwise: transfer to step 1.

In more detail:

Starting from cell (1, 1), is given to the variable x_{ij} the maximum value which either satisfies the needs of destination j or exhausts the remaining capacity of source I, and particularly the smaller of these two quantities. Then a value is given to the variable x_{ij+1} in the former case or to the variable x_{i+1j} in the latter. Due to the properties of the transportation problem, the value of the last variable x_{mn} is such that the capacity of source m and the need of destination n are satisfied simultaneously.

The Northwest Corner Method is simple in its use, however does not utilize shipping costs. It can yield an initial basic feasible solution easily, but the corresponding total shipping cost may be very high.

Initial feasible solution with the Northwest corner method

	K_1	K_2	K_3	K_4	K_5	
L_1	75 3	 2	 3	 4	 1	75
L_2	25 4	60 1	40 2	25 4	 2	150
L_3	 1	 0	 5	50 3	25 2	75
	100	60	40	75	25	

K_{min} = € 765

Minimum cost method

The minimum cost method uses shipping costs in order to come up with a basic feasible solution that has lower total cost. To begin the method, first the variable x_{ij} with the smallest shipping cost is located. The largest possible value is assigned to variable x_{ij}; this value is the minimum of a_i and b_j.

After that, as in the Northwest Corner Method, row i or column j is crossed out and the supply or the demand of the non-crossed out row or column is reduced by the value of x_{ij}. The next route (cell) with the minimum shipping cost is chosen among the ones which do not belong to the crossed-out row or column. This procedure is repeated until all capacities are exhausted and all demands are satisfied.

Initial feasible solution with the minimum cost method

	K_1	K_2	K_3	K_4	K_5	
L_1	50 3	2	3	4	25 1	75
L_2	35 4	1	40 2	75 4	2	150
L_3	15 1	60 0	5	3	2	75
	100	60	40	75	25	

K_{min} = € 710

Vogel method

Methodology steps

1. Addition - below and right of the transportation tableau – of a new row and a new column with elements the difference of the two smaller cost elements of each row and each column respectively.

2. Selection of the largest element of the added two new lines.

3. Finding of the minimum element of row i or column j in which belongs the element identified in step 2.

4. Assignment of the value $x_{ij} = \min(a_i, b_j)$ to the route corresponding to the position of the smallest element in order to meet the capacity of a source or the demand of a destination.

5. If the capacity of a source is exhausted, the demand b_j of the corresponding destination is reduced by a_i. In contrary, if the demand of a destination is satisfied, the capacity a_i of the corresponding source is reduced by b_j. The source (row) or destination (column) that was satisfied is crossed-out and is not taken further into account.

Each time the above procedure is repeated the capacity of a source is exhausted or the needs of a destination are satisfied. The implementation of the method is completed when the capacity of the last row and the needs of the last column are simultaneously satisfied. The solution yielded is feasible because it meets all capacities and all needs.

Successive steps of Vogel method

1^{st} step

	K_1	K_2	K_3	K_4	K_5	**Difference**
L_1	3	2	3	4	1	1
L_2	4	1	2	4	2	1
L_3	1	0	5	3	2	1
Difference	2	1	1	1	1	

	K_1	K_2	K_3	K_4	K_5		
L_1							75
L_2							150
L_3	75					~~75~~	0
	~~100~~ 25	60	40	75	25		

2^{nd} step

	K_1	K_2	K_3	K_4	K_5	**Difference**
L_1	3	2	3	4	1	1
L_2	4	1	2	4	2	1
L_3						
Difference	1	1	1	0	1	

	K_1	K_2	K_3	K_4	K_5		
L_1					25	~~75~~	50
L_2							150
L_3	75						0
	25	60	40	75	~~25~~ 0		

3^{rd} step

	K_1	K_2	K_3	K_4	K_5	Difference
L_1	3	2	3	4		1
L_2	4	1	2	4		1
L_3						
Difference	1	1	1	0		

	K_1	K_2	K_3	K_4	K_5	
L_1	25				25	~~50~~ 25
L_2						150
L_3	75					0
	~~25~~ 0	60	40	75	0	

4^{th} step

	K_1	K_2	K_3	K_4	K_5	Difference
L_1		2	3	4		1
L_2		1	2	4		1
L_3						
Difference		1	1	0		

	K_1	K_2	K_3	K_4	K_5	
L_1	25	25			25	~~25~~ 0
L_2						150
L_3	75					0
	0	~~60~~ 35	40	75	0	

5^{th} cycle

	K_1	K_2	K_3	K_4	K_5	Difference
L_1						
L_2		1	2	4		1
L_3						
Difference		-	-	-		

	K_1	K_2	K_3	K_4	K_5	
L_1	25	25			25	25
L_2		35				~~150~~ 115
L_3	75					0
	0	~~35~~ 0	40	75	0	

6^{th} step

	K_1	K_2	K_3	K_4	K_5	**Difference**
L_1						
L_2			2	4		1
L_3						
Difference			-	-		

	K_1	K_2	K_3	K_4	K_5		
L_1	25	25			25		25
L_2		35	40			~~115~~	75
L_3	75						0
	0	0	~~40~~ 0	75	0		

Initial basic feasible solution (Vogel method)

	K_1	K_2	K_3	K_4	K_5	
L_1	25 3	 2	 3	 4	25 1	75
L_2	 4	35 1	40 2	75 4	 2	150
L_3	75 1	 0	 5	 3	 2	75
	100	60	40	75	25	

K_{min} = € 640

Degenerate initial solution

	K_1	K_2	K_3	K_4	K_5	
L_1	100 3	2	3	4	1	100
L_2	4	60 1	40 2	25 4	2	125
L_3	1	0	5	50 3	25 2	75
	100	60	40	75	25	

Initial solution that is no longer degenerate

	K_1	K_2	K_3	K_4	K_5	
L_1	100 3	ε 2	 3	 4	 1	100+ε
L_2	 4	60 1	40 2	25 4	 2	125
L_3	 1	 0	 5	50 3	25 2	75
	100	60+ε	40	75	25	

Transportation problems solution methodology

Initial step

Creation of an initial basic feasible solution using one of the three relevant methods.
Transfer to the termination rule.

Repeating step

1. Determination of the incoming variable that will be introduced to the base:
   Selection of the non basic variable x_{ij} with maximum negative difference $c_{ij} - u_i - v_j$

2. Determination of the outgoing variable that will be reTransferred from the base:
   Identification of the (unique) loop, which's vertices are only basic variables.
   Assignment to the incoming variable of the maximum possible value.
   For its determination it is selected among the donor routes the one with the smallest value. The corresponding variable is reTransferred from the basis.

3. Determination of the new basic feasible solution:
   Addition of quantity θ to each recipient route and deduction from each donor route, so that all constraints of sources and destinations are met.

Termination rule

Calculation of the elements u_i and v_j.

[It is suggested to select the line (row or column) with the largest number of basic variables, give to the corresponding u_i (v_j) the zero value and solve the system of equations $c_{ij} = u_i + v_j$ for each basic route(i, j)]

Solution optimality control

If for each non basic route (i,j) $u_i + v_j \le c_{ij}$ then the solution is optimal ➔ End.

In not, go to the repetitive step.

Successive steps of finding the optimal solution

1^{st} repetition

		K_1	K_2	K_3	K_4	K_5	
	v_j / u_i	4	1	2	4	3	
L_1	-1	75 3	2	3	4	* 1	75
L_2	-0	25-θ 4	60 1	40 2	25+θ 4	* 2	150
L_3	-1	θ * 1	0	5	50-θ 3	25 2	75
		100	60	40	75	25	

The largest quantity that can be transferred from route L2 - K1 (outgoing variable) to the route L3 - K1 (incoming variable as $c_{31} = u_3 - v_1 = 1-(-1)-4 = -2 < 0$) is $\theta_{max} = 25$.

2^{nd} repetition

		K_1	K_2	K_3	K_4	K_5	
	v_j / u_i	2	1	2	4	3	
L_1	1	75-θ 3	 2	 3	* 4	θ * 1	75
L_2	-0	 4	60 1	40 2	50 4	* 2	150
L_3	-1	25+θ 1	 0	 5	25 3	25-θ 2	75
		100	60	40	75	25	

Total transportation cost: 715

Incoming variable : x_{15} ($c_{15}-u_1-v_5 = -1$)

Outgoing variable : x_{35}

Transferred quantity : $\theta_{max} = 25$

3^{rd} repetition

		K_1	K_2	K_3	K_4	K_5	
	v_j \ u_i	2	1	2	4	3	
L_1	1	50-θ 3	2	3	θ * 4	25 1	75
L_2	-0	4	60 1	40 2	50 4	2	150
L_3	-1	50+θ 1	0	5	25-θ 3	2	75
		100	60	40	75	25	

Total transportation cost : 640 (equal to the solution of Vogel method)
Incoming variable : x_{14} ($c_{14}-u_1-v_4 = -1$)
Outgoing variable : x_{34}
Transferred quantity : $\theta_{max} = 25$

4^{th} repetition

		K_1	K_2	K_3	K_4	K_5	
	v_j / u_i	2	1	2	4	3	
L_1	1	25 3	2	3	25 4	25 1	75
L_2	-0	4	60 1	40 2	50 4	2	150
L_3	-1	75 1	0	5	3	2	75
		100	60	40	75	25	

This solution is optimal as for each non basic variable x_{ij} the condition $u_i + v_j \leq c_{ij}$ is valid. The minimum total transportation cost is € 615.

Thus, the optimal transportation and distribution program of the 300 tones of gravel is the following:

from mine L_1
- → 25 tones to consumer K_1
- → 25 - " - K_4
- → 25 - " - K_5

from mine L_2
- → 60 tones to consumer K_1
- → 40 - " - K_3
- → 50 - " - K_4

from mine L_3 → 75 tones to consumer K_1

Degenerate solution

	K_1	K_2	K_3	K_4	K_5	
L_1	100 3	E 2	3	4	1	100+ε
L_2	4	60 1	40 2	25 4	2	125
L_3	θ 1	0	5	50 3	25 2	75
	100	60+ε	40	75	25	

1^{st} repetition

		K_1	K_2	K_3	K_4	K_5	
	v_j \ u_i	2	1	2	4	3	
L_1	1	100 3	ε-θ 2	 3	* 4	θ* 1	100+ε
L_2	0	 4	60-θ 1	40 2	25-θ 4	* 2	125
L_3	-1	 1	 0	 5	50-θ 3	25-θ 2	75
		100	60+ε	40	75	25	

With asterisk (*) are marked all the routes with negative difference c_{ij}-u_i-v_j

Total transportation cost	:	740
Incoming variable	:	x_{15} ($c_{15}-u_1-v_5 = -3$)
Outgoing variable	:	x_{12}
Transferring quantity	:	$\theta_{max} = \varepsilon$

2^{nd} repetition

		K_1	K_2	K_3	K_4	K_5	
	v_j / u_i	5	1	2	4	3	
L_1	1	100-θ 3	2	3	4	ε+θ 1	100+ε
L_2	0	* 4	60+ε 1	40 2	50-ε 4	* 2	125
L_3	-1	θ* 1	0	5	50+ε 3	25-ε-θ 2	75
		100	60+ε	40	75	25	

Total transportation cost	:	740
Incoming variable	:	x_{31} (c_{31}-u_3-v_1 = -1)
Outgoing variable	:	x_{35}
Transferring quantity	:	θ_{max} = 25-ε

3^{rd} repetition

		K_1	K_2	K_3	K_4	K_5	
	v_j / u_i	2	1	2	4	3	
L_1	1	75-θ 3	2	3	θ* 4	25 1	100+ε
L_2	0	4	60+ε 1	40 2	25-ε 4	2	125
L_3	-1	25-ε+θ 1	0	5	50+ε-θ 3	2	75
		100	60+ε	40	75	25	

Total transportation cost	:	665
Incoming variable	:	x_{14} ($c_{14}-u_1-v_4 = -1$)
Outgoing variable	:	x_{34}
Transferring quantity	:	$\theta_{max} = 50+\varepsilon$

4^{th} repetition

		K_1	K_2	K_3	K_4	K_5	
	v_j / u_i	3	1	2	4	1	
L_1	0	25 3	 2	 3	50+ε 4	25 1	100+ε
L_2	0	 4	60+ε 1	40 2	25-ε 4	 2	150
L_3	-1	75 1	 0	 5	 3	 2	75
		100	60+ε	40	75	25	

For all non basic routes of the last table the condition $u_i + v_j \leq c_{ij}$ is valid, so the current solution is optimal. In order to identify this solution the auxiliary variable ε is deleted, having completed its contribution. The total transportation cost (€ 615) and the analytical program of transportation and distribution of 300 tones of gravel can now be determined.

Optimal solution

	K_1	K_2	K_3	K_4	K_5	
L_1	25 3	 2	 3	50 4	25 1	100
L_2	 4	60 1	40 2	25 3	 2	125
L_3	75 1	 0	 5	 3	 2	75
	100	60	40	75	25	

Lack of balance

1. Surplus production

$$\sum_{j=1}^{m} x_{ij} \leq a_i$$ (where i = 1,2,...,m)

$$\sum_{i=1}^{n} x_{ij} \leq b_i$$ (where j = 1,2,...,n)

$$\sum_{j=i}^{m} a_i \geq \sum_{j=i}^{n} b_i$$

Creation of a fictitious destination (demand location)

The values of the additional column’s cells depend on the type and characteristics of the specific problem.

2. Deficient production

$$\sum_{j=i}^{m} x_{ij} = a_i$$ (where i = 1,2,...,m)

$$\sum_{i=i}^{n} x_{ij} \leq b_i$$ (where j = 1,2,...,n)

$$\sum_{j=i}^{m} a_i \leq \sum_{j=i}^{n} b_i$$

Creation of a fictitious source (supply location)

Regarding the elements of the additional row:

- If the shortage cost of a quantity that has to be transported to a destination is zero, the corresponding cost element is equal to zero.

- If the inability to meet the demand implies some economic consequences (penalties, discounts, cost of good reputation, etc), then the cost of each cell of the additional row is put equal to the corresponding unit shortage cost.

3. Satisfaction obligation

In such cases the transportation tableau has to be formed in such a way that the corresponding additional route of the fictitious source or destination will not participate in any case in the final solution. For this purpose, a very large positive value (M) is given to the cost element of this route. This ensures that this route will not participate in any case in the final solution.

4. Surplus or deficient production

It is possible that in some transportation problems is not known in advance whether the production will be surplus or deficient. This can happen if the elements c_{ij} of the objective function represent results (profit, loss or another efficiency criterion) from the satisfaction of demand location. Some of these elements may be negative. Therefore, it may be preferable not to satisfy at all a demand than the corresponding economic result to lead in loss.

In order to address such a situation a fictitious source (row) and a fictitious destination (column) are added to the initial transportation table. The capacity and demand of the two additional lines should be such as neither the satisfaction of all demands or the exhaustion of all capacities is compulsory.

Solution of maximization problems

In such a case the target is the maximization of the objective function. The methodology of finding the optimal solution of such a problem is nearly identical to that of minimization problems. The steps after the determination of an initial feasible solution remain unchanged. The only difference refers to the termination rule examining the optimality of the solution:

The current solution is optimal if for any non-basic route is valid the relation:

$\mathbf{u_i + v_j \geq c_{ij}}$

Differences in the methods of finding an initial solution

- Northwest corner method: No difference.

- Minimum cost method: In practice I renamed to “maximum profit method”.
  So, keeping the same logic, the priority is the assignment of the maximum possible quantity to the route with the highest unit profit.

- Vogel method: The differences, due to its complexity, are more:
  - Step 1: The elements of the additional column and row are the differences between the two larger profit elements of each row and column respectively.
  - Step 3: The largest element of the appropriate column or row is searched out in the transportation table.

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